Toulouse School of Economics

M1 INTERMEDIATE ECONOMETRICS

Large-sample asymptotics

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Overview

This deck of slides goes through the basics of asymptotic theory.

The corresponding chapter in Hansen is 6.

Convergence in probability

Consider a sequence of random vectors $Z_1, Z_2, \ldots, Z_n$.

Z_n converges in probability to Z as $n \to \infty$ if for all constants $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}(\|Z_n - Z\| \leq \delta) = 1.$$

The random variable Z is called the probability limit of Z_n.

We write $Z_n \underset{p}{\to} Z$ or $\text{plim}_{n\to\infty} Z_n = Z$.

Examples

Consider binary Z_n with $\mathbb{P}(Z_n = 0) = 1 - p_n$ and $\mathbb{P}(Z_n = 1) = p_n$.

Suppose that $p_n \to 0$ as $n \to \infty$:

For any $\delta \geq 1$, $\mathbb{P}(|Z_n| \leq \delta) = 1$,

For any $0 < \delta < 1$, $\mathbb{P}(|Z_n| \leq \delta) = \mathbb{P}(Z_n = 0) = 1 - p_n \to 1$,

and so $Z_n \underset{p}{\to} 0$.

Consider binary Z_n with $\mathbb{P}(Z_n = 0) = 1 - p$ and $\mathbb{P}(Z_n = a_n) = p$.

Suppose that $a_n \to 0$ as $n \to \infty$:

As $a_n \to 0$, for any $\delta > 0$ there exists a value for n at which $a_n \leq \delta$. Hence, $Z_n \underset{p}{\to} 0$.

Let Z be a random variable independent of n and let

$$Y_n = \begin{cases} 1 & \text{with probability } 1/n \\ 0 & \text{with probability } 1 - 1/n \end{cases} .$$

Define $Z_n = Z + n\,Y_n$.

To show that $Z_n \underset{p}{\rightarrow} Z$ note that $Z_n - z = n\,Y_n$. Hence, for sufficiently small δ

$$\mathbb{P}(|Z_n - Z| > \delta) = \mathbb{P}(n\,Y_n > \delta) = \mathbb{P}(Y_n = 1) = 1/n,$$

which goes to zero as $n \to \infty$.

Suppose that

$$Z_n \sim N(0, {}^{\sigma^2}\!/_{a_n^2}).$$

for some $a_n \to \infty$ as $n \to \infty$.

Then

$$\mathbb{P}(|Z_n - 0| > \delta) = \mathbb{P}(Z_n \leq -\delta) + \mathbb{P}(Z_n > \delta)$$

with

$$\mathbb{P}(Z_n \leq -\delta) = \Phi\left(-a_n {}^{\delta}\!/_{\sigma}\right),$$

and, by symmetry of the normal distribution,

$$\mathbb{P}(Z_n > \delta) = \mathbb{P}(Z_n \leq -\delta) = \Phi\left(-a_n {}^{\delta}\!/_{\sigma}\right).$$

Thus,

$$\mathbb{P}(|Z_n - 0| > \delta) = 2\,\Phi\left(-a_n {}^{\delta}\!/_{\sigma}\right) \underset{n\uparrow\infty}{\rightarrow} 0$$

Now suppose that

$$Z_n \sim N(0, \sigma_n^2),$$

and let a_n be some other deterministic sequence that grows with n.

Then

$$\mathbb{P}(a_n|Z_n - 0| > \delta) = 2\,\Phi\left(-{}^{\delta}/_{a_n\sigma_n}\right).$$

This goes to zero provided that

$$a_n\sigma_n \to 0.$$

When $a_n\sigma_n$ converges to a finite constant $c > 0$ we have that

$$\mathbb{P}(a_n|Z_n - 0| > \delta) = 2\,\Phi(-{}^{\delta}/_{c})$$

as then $a_n Z_n \sim N(0, c^2)$.

We say that

$$Z_n = o_p(a_n)$$

when $a_n^{-1} Z_n \underset{p}{\rightarrow} 0$

We say that

$$Z_n = O_p(a_n)$$

if and only if for every ε there exists a finite number M_ε and an n_ε^* such that

$$\mathbb{P}(a_n^{-1} \|Z_n\| > M_\varepsilon) \leq \varepsilon$$

for all $n \geq n_\varepsilon^*$.

Then $a_n^{-1} Z_n = O_p(1)$, that is, the sequence $a_n^{-1} Z_n$ is stochastically bounded.

Continuous mapping theorem

Let $Z_n \underset{p}{\rightarrow} c$ for some constant c.

Let g be a function that is continuous at c.

Then $g(Z_n) \underset{p}{\rightarrow} g(c)$.

Sample-mean theorem

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample on Y. Suppose that Y has finite mean μ and variance σ^2.

Then the sample mean

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

satisfies

$$\mathbb{E}(\bar{Y}) = \mu, \qquad \mathrm{var}(\bar{Y}) = \frac{\sigma^2}{n}.$$

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample on Y. Suppose that Y has finite mean μ.

Then

$$\bar{Y} \underset{p}{\rightarrow} \mu.$$

An implication is that, for any function $h(Y)$ with finite mean, we have that

$$\frac{1}{n}\sum_{i=1}^{n} h(Y_i) \underset{p}{\rightarrow} \mathbb{E}(h(Y)).$$

Under the additional condition that $\mathrm{var}(Y) = \sigma^2 < \infty$ (for the scalar case) we have

$$\mathbb{P}(|\bar{Y} - \mu| > \delta) \leq \frac{\mathbb{E}[(\bar{Y} - \mu)^2]}{\delta^2} = \frac{\mathrm{var}(\bar{Y})}{\delta^2} = \frac{1}{n}\,\frac{\sigma^2}{\delta^2}$$

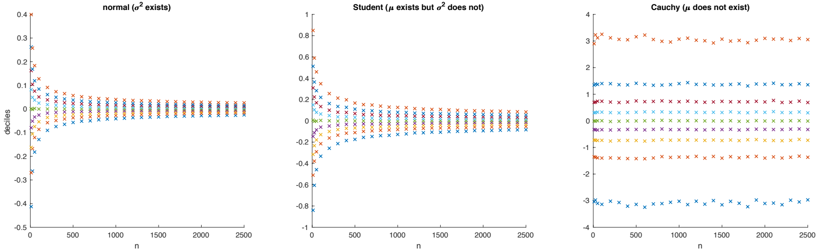
which converges to zero as $n \to \infty$.

Here,

the first step follows from Markov's (Chebychev's) inequality, and

the second step follows from the sample-mean theorem.

The below plots give deciles of the distribution of $\overline{Y}$ as a function of n.



Note that

$$\sqrt{n}\,(\bar{Y}-\mu) = O_p(1)$$

or, equivalently,

$$\bar{Y}-\mu = O_p(n^{-1/2}).$$

Indeed,

$$\mathbb{P}(\sqrt{n}|\bar{Y}-\mu| > \delta) \leq \frac{\mathbb{E}[n\,(\bar{Y}-\mu)^2]}{\delta^2} = \frac{n\,\mathrm{var}(\bar{Y})}{\delta^2} = \frac{\sigma^2}{\delta^2}.$$

Consequently,

$$n^a(\bar{Y}-\mu) \underset{p}{\to} 0$$

for any $a < 1/2$, that is, $n^a(\bar{Y}-\mu) = o_p(1)$ for any such a.

On the other hand, for any $a > 1/2$,

$$n^a(\bar{Y}-\mu) = n^{a-1/2}n^{1/2}(\bar{Y}-\mu) = n^{a-1/2}O_p(1) = O_p(n^{a-1/2})$$

diverges as $n \to \infty$.

Convergence in distribution

Let $Z_n \sim F_n$ and $Z \sim F$.

Z_n converges in distribution to Z as $n \to \infty$ if

$$F_n(z) \to F(z)$$

holds at all continuity points z of F as $n \to \infty$. F is called the limit distribution of Z_n.

We write $Z_n \underset{d}{\to} Z$.

Examples

Let Z_z have mixture distribution

$$F_n(z) = \Phi(z)\, p_n + \Phi(z-1)\,(1-p_n).$$

If $p_n \to 1$ as $n \to \infty$ then

$$F_n(z) \to \Phi(z)$$

for all z and so $Z_n \underset{d}{\to} Z$ for $Z \sim N(0,1)$.

If $p_n \to 0$ as $n \to \infty$ then

$$F_n(z) \to \Phi(z-1)$$

for all z and so $Z_n \underset{d}{\to} Z$ for $Z \sim N(1,1)$.

If $p_n \to p \in (0,1)$ then $F_n(z) \to \Phi(z)\, p + \Phi(z-1)\,(1-p)$.

Let $Z_n \underset{d}{\to} Z$.

Let g be a function that is continuous in Z (with probability one).

Then $g(Z_n) \underset{d}{\to} g(Z)$.

Let $Y_1, \ldots, Y_n$ be a random sample on Y.

If $\mathbb{E}(\|Y\|^2) < \infty$ with

$$\mu = \mathbb{E}(Y), \qquad V = \mathrm{var}(Y),$$

then

$$\sqrt{n}(\bar{Y} - \mu) \underset{d}{\rightarrow} N(0, V)$$

as $n \to \infty$.

Alternatively, if V is non-singular, then

$$\sqrt{n}\, V^{-1/2}(\bar{Y} - \mu) \underset{d}{\rightarrow} N(0, I)$$

as $n \to \infty$.

The central limit theorem broadly means that, in large samples, sample averages are 'close to' being normally distributed.

Moreover,

$$\bar{Y} = \mu + \frac{1}{\sqrt{n}} V^{1/2} Z + o_p(n^{-1/2})$$

for $Z \sim N(0, I)$.

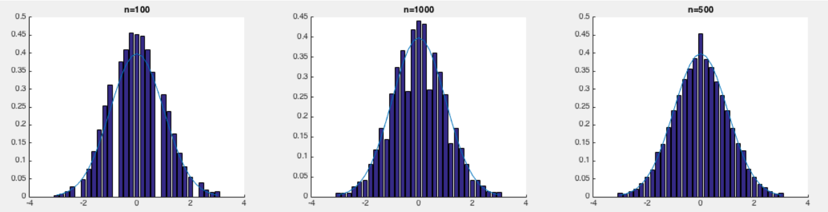
That is,

$$\bar{Y} \underset{a}{\sim} N(\mu, V/n),$$

where $\underset{a}{\sim}$ can be interpreted as 'approximately distributed as'.

The plots below concern the standardized sample mean of samples of Bernoulli random variables.

Observe how the histogram approaches the standard-normal density as n grows.



Let $c = (c_1, \ldots, c_k)' \in \mathbb{R}^k$.

Let $\sqrt{n}(Z_n - c) \underset{d}{\to} N(0, V)$ as $n \to \infty$.

Let $g = (g_1, \ldots, g_q)' : \mathbb{R}^k \to \mathbb{R}^q$ be continuously differentiable in a neighbourhood of c.

The $k \times q$ Jacobian is

$$(G(u))_{i,j} = \frac{\partial g_j(u)}{\partial u_i}.$$

We write $G = G(c)$.

Then

$$\sqrt{n}(g(Z_n) - g(c)) \underset{d}{\to} N(0, G'VG)$$

as $n \to \infty$.

Examples

Take

$$\sqrt{n}(\bar{Y} - \mu) \underset{d}{\rightarrow} N(0, \sigma^2).$$

For $g(u) = \exp(u)$, $G(u) = \exp(u)$, and so

$$\sqrt{n}(\exp(\bar{Y}) - \exp(\mu)) \underset{d}{\rightarrow} N(0, \exp(\mu)^2\sigma^2).$$

For $g(u) = u^3$, $G(u) = 3u^2$, and so

$$\sqrt{n}(\bar{Y}^3 - \mu^3) \underset{d}{\rightarrow} N(0, 9\mu^4\sigma^2).$$

Take

$$\sqrt{n}\begin{pmatrix} \bar{X}-\mu_X \\ \bar{Y}-\mu_Y \end{pmatrix} \underset{d}{\rightarrow} N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{matrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{matrix} \right).$$

Suppose that $\mu_Y \neq 0$.

Then for

$$g(\mu_X, \mu_Y) = \frac{\mu_X}{\mu_Y}$$

we have

$$G = \begin{pmatrix} {}^{1}\!/_{\mu_Y} \\ -{}^{\mu_X}\!/_{\mu_Y^2} \end{pmatrix}$$

and so

$$\sqrt{n}\left({}^{\bar{X}}\!/_{\bar{Y}} - {}^{\mu_X}\!/_{\mu_Y}\right)$$

is asymptotically normal with variance

$$\begin{pmatrix} {}^{1}\!/_{\mu_Y} & -{}^{\mu_X}\!/_{\mu_Y^2} \end{pmatrix} \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \begin{pmatrix} {}^{1}\!/_{\mu_Y} \\ -{}^{\mu_X}\!/_{\mu_Y^2} \end{pmatrix}$$

Suppose that

$$X_n \underset{p}{\to} c, \qquad Y_n \underset{d}{\to} Y,$$

as $n \to \infty$.

Then

$X_n + Y_n \underset{d}{\to} c + Y,$

$X_n Y_n \underset{d}{\to} cY.$

Example

Suppose that scalar random variable Y has finite mean μ and variance σ^2.

Then

$$\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \underset{d}{\rightarrow} N(0,1)$$

by the central limit theorem.

Also,

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2} \underset{p}{\rightarrow} \sigma$$

To see this note first that

$$\frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \mu)^2 - (\bar{Y} - \mu)^2.$$

Here,

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu)^2 \underset{p}{\to} \sigma^2$$

by the law of large numbers, and $(\bar{Y}-\mu)^2 \underset{p}{\to} 0$ because $\bar{Y} \underset{p}{\to} \mu$.

Hence,

$$s^2 = \frac{n}{n-1}\,\frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2 \underset{p}{\to} \sigma^2$$

and, by the continuous mapping theorem, $s \underset{p}{\to} \sigma$.

Therefore,

$$\sqrt{n}\frac{\bar{Y}-\mu}{s} = \frac{\sigma}{s}\,\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} = (1+o_p(1))\ \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \underset{d}{\to} N(0,1)$$

by Slutzky's theorem.